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Generalized instantons in $\mathcal{N} = 4$ super Yang-Mills theory and spinorial geometry

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Abstract: Using spinorial geometry techniques, we classify the supersymmetric solutions of euclidean $\mathcal{N} = 4$ super Yang-Mills theory. These backgrounds represent generalizations of instantons with nontrivial scalar fields turned on, and satisfy some constraints that bear a similarity with the Hitchin equations, and contain the Donaldson equations as a special subcase. It turns out that these constraints can be obtained by dimensional reduction of the octonionic instanton equations, and may be rephrased in terms of a selfduality-like condition for a complex connection. We also show that the supersymmetry conditions imply the equations of motion only partially.

Keywords: Supersymmetric gauge theory, Solitons Monopoles and Instantons, AdS-CFT Correspondence

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Contents

1 Introduction

Supersymmetric backgrounds play an important role both in supergravity and in supersymmetric gauge theories. In the former, they include for instance BPS black holes, whose study has given many hints on the quantum nature of spacetime (see e.g. [1]). In the latter, there are e.g. monopoles or instantons among the supersymmetric solutions. Instantons are also important in nonsupersymmetric field theories, like in QCD, where the nonperturbative chiral U(1) anomaly in an instanton background leads to baryon number violation and to a solution of the U(1) problem [2, 3]. Viewed as a solution of supersymmetric gauge theories, the Yang-Mills instanton [4] preserves half of the supersymmetries, and has been important in checks of the AdS/CFT correspondence beyond the perturbative level, cf. [5] for a review.

In view of this, it is desirable to dispose of a complete classification of supersymmetric backgrounds of super Yang-Mills theories. This paper represents a first step in this direction for euclidean $\mathcal{N} = 4$ super Yang-Mills theory, which is the most important one in the context of the AdS/CFT correspondence. We shall directly solve the Killing spinor equations using spinorial geometry techniques, that have been successfully applied in the past in classifying supergravity solutions [6]. The basic ingredients are an oscillator basis for the spinors in terms of forms and the use of the symmetries to transform them to a preferred representative of their orbit. In this way one can construct a linear system for the background fields from any (set of) Killing spinor(s). It will turn out that this linear system describes generalizations of instantons that include also nonvanishing scalars. Quite remarkably, this system can be obtained by dimensional reduction of the octonionic instanton equations in eight dimensions.

The remainder of this paper is organized as follows: In the next section, we introduce euclidean $\mathcal{N} = 4$ super Yang-Mills theory and, to make this paper self-contained, briefly discuss the usual instanton solutions. In section 3, the essential information needed to realize spinors in terms of forms is summarized. Section 4 represents the main part of this work, in which we determine first the Killing spinor representatives as well as their stability subgroup, and subsequently obtain the linear system for the background fields, which is then discussed and related to the octonionic instanton equations. After that, we impose more Killing spinors and investigate which fractions of supersymmetry are possible. In section 5 we discuss to what extent the supersymmetry conditions imply the equations of motion. We conclude in 6 with some final remarks. An appendix contains our notations and conventions.

2 Euclidean $\mathcal{N} = 4$ SYM theory and instantons

The Lagrangian of euclidean $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions is [7, 8]

$$\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\left\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}^{\dot{\alpha}}_{A}\bar{\not{D}}_{\dot{\alpha}\beta}\lambda^{\beta,A} - i\lambda^{A}_{\alpha}\not{D}^{\alpha\dot{\beta}}\bar{\lambda}_{\dot{\beta},A} + \frac{1}{2}\left(D_\mu\bar{\phi}_{AB}\right)\left(D^\mu\phi^{AB}\right) \right.$$
$$\left. - \sqrt{2}\bar{\phi}_{AB}\left\{\lambda^{\alpha,A},\lambda^{B}_{\alpha}\right\} - \sqrt{2}\phi^{AB}\left\{\bar{\lambda}^{\dot{\alpha}}_{A},\bar{\lambda}_{\dot{\alpha},B}\right\} + \frac{1}{8}\left[\phi^{AB},\phi^{CD}\right]\left[\bar{\phi}_{AB},\bar{\phi}_{CD}\right]\right\}. \tag{2.1}$$

It can be obtained by dimensionally reducing $\mathcal{N} = 1$ SYM in 10-dimensional Minkowski space-time on a six-torus with one time and five space coordinates [8–10].[1] The ten-dimensional Lorentz group $\mathrm{SO}(9,1)$ reduces then to $\mathrm{SO}(4)\times\mathrm{SO}(5,1)$, with compact space-time group $\mathrm{SO}(4)$ and non-compact R-symmetry $\mathrm{SO}(5,1)$. The fields present are the gauge field A_μ with field strength $F_{\mu\nu}$, $\mathrm{Spin}(4)$ Weyl spinors $\lambda^{\alpha,A}$ and $\bar{\lambda}_{\dot{\alpha},A}$, respectively right- and left-handed, and six scalars ϕ^{AB} antisymmetric in the R-symmetry group indices $A, B = 1,\ldots,4$. Their duals are defined as $\bar{\phi}_{AB} = \frac{1}{2}\epsilon_{ABCD}\phi^{CD}$. The trace is taken in the adjoint representation of the gauge group to which all the fields belong (the gauge group indices are omitted for clarity). The Majorana-Weyl condition imposed on the ten-dimensional spinors implies the so-called symplectic Majorana condition on the four-dimensional Weyl

[1] Note that one cannot get the theory (2.1) by simply Wick-rotating Minkowskian $\mathcal{N} = 4$ SYM. In fact, the Wick rotation of the bosonic sector of $\mathcal{N} = 4$ SYM theory has no supersymmetric completion. See [11] for a discussion. Notice also that in (2.1), the scalar coming from the time component A_0 of the ten-dimensional vector potential has a kinetic term of the wrong sign, so the theory has ghosts. It is easy to see that this (real) scalar is $i(\phi^{14} + \phi^{23})$.

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spinors[2] (the usual Majorana condition cannot be imposed in four euclidean dimensions):

$$\begin{aligned}\left(\lambda^{\alpha,A}\right)^* &= -\eta_{1AB}\epsilon_{\alpha\beta}\lambda^{\beta,B} = \eta_{1AB}\lambda^B_\alpha \\ \left(\bar\lambda_{\dot\alpha,A}\right)^* &= -\eta_1^{AB}\epsilon^{\dot\alpha\dot\beta}\bar\lambda_{\dot\beta,B} = -\eta_1^{AB}\bar\lambda^{\dot\alpha}_B\,,\end{aligned}\tag{2.2}$$

where the definition of the 't Hooft symbols η_{aAB} and other conventions are collected in appendix A. The scalar fields on the other hand are constrained by the reality condition

$$\left(\phi^{AB}\right)^* = \eta_{1AC}\phi^{CD}\eta_{1DB}\,,\tag{2.3}$$

or explicitly

$$(\phi^{12})^* = \phi^{34}\,,\qquad (\phi^{13})^* = -\phi^{24}\,,\qquad (\phi^{14})^* = -\phi^{14}\,,\qquad (\phi^{23})^* = -\phi^{23}\,,\tag{2.4}$$

since they originate from the dimensional reduction of the real 10-dimensional gauge field.

The Lagrangian (2.1) is invariant under the supersymmetry transformations

$$\begin{aligned}\delta A_\mu &= -i\bar\xi^{\dot\alpha}_A\bar\sigma_{\mu\dot\alpha\beta}\lambda^{\beta,A} + i\bar\lambda_{\dot\beta,A}\sigma^{\alpha\dot\beta}_\mu\xi^A_\alpha\,,\\ \delta\phi^{AB} &= \sqrt{2}\left(\xi^{\alpha,A}\lambda^B_\alpha - \xi^{\alpha,B}\lambda^A_\alpha + \epsilon^{ABCD}\bar\xi^{\dot\alpha}_C\bar\lambda_{\dot\alpha,D}\right)\,,\\ \delta\lambda^{\alpha,A} &= -\frac{1}{2}\sigma^{\mu\nu\alpha}{}_\beta F_{\mu\nu}\xi^{\beta,A} - i\sqrt{2}\bar\xi_{\dot\alpha,B}\not{D}^{\alpha\dot\alpha}\phi^{AB} + \left[\phi^{AB},\bar\phi_{BC}\right]\xi^{\alpha,C}\,,\\ \delta\bar\lambda_{\dot\alpha,A} &= -\frac{1}{2}\bar\sigma^{\mu\nu}{}_{\dot\alpha}{}^{\dot\beta}F_{\mu\nu}\bar\xi_{\dot\beta,A} + i\sqrt{2}\xi^{\alpha,B}\not{D}_{\dot\alpha\alpha}\bar\phi_{AB} + \left[\bar\phi_{AB},\phi^{BC}\right]\bar\xi_{\dot\alpha,C}\,,\end{aligned}\tag{2.5}$$

whose fermionic parameters ξ^A and $\bar\xi_A$ themselves have to satisfy the symplectic Majorana condition (2.2). The equations of motion derived from (2.1) read

$$D^\nu F_{\nu\mu} - i\left\{\bar\lambda^{\dot\alpha}_A\bar\sigma_{\mu\dot\alpha\beta},\lambda^{\beta,A}\right\} - \frac{1}{2}\left[\bar\phi_{AB},D_\mu\phi^{AB}\right] = 0\,,\tag{2.6}$$

$$D^2\phi^{AB} + \sqrt{2}\left\{\lambda^{\alpha,A},\lambda^B_\alpha\right\} + \frac{1}{\sqrt{2}}\epsilon^{ABCD}\{\bar\lambda^{\dot\alpha}_C,\bar\lambda_{\dot\alpha,D}\} - \frac{1}{2}\left[\bar\phi_{CD},\left[\phi^{AB},\phi^{CD}\right]\right] = 0\,,\tag{2.7}$$

$$\bar{\not{D}}_{\dot\alpha\beta}\lambda^{\beta,A} + i\sqrt{2}\left[\phi^{AB},\bar\lambda_{\dot\alpha,B}\right] = 0\,,\tag{2.8}$$

$$\not{D}^{\alpha\dot\beta}\bar\lambda_{\dot\beta,A} - i\sqrt{2}\left[\bar\phi_{AB},\lambda^{\alpha,B}\right] = 0\,.\tag{2.9}$$

A notable solution of the pure euclidean Yang-Mills field equations is given by a field strength which is either selfdual or anti-selfdual,

$$F_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,.\tag{2.10}$$

These solutions correspond to (anti-)instantons, i.e. finite-action solutions to the euclidean theory. In a given topological sector (characterized by the instanton or winding number k), the solutions (2.10) actually minimize the action (for reviews on instantons, see e.g. [5, 8, 12, 13]). These configurations also generically correspond to solutions preserving some fraction of the supersymmetries in SYM theories. In the $\mathcal{N} = 2$ case for instance (see

[2]Unless specified otherwise, equations which involve complex conjugation of fields will be understood as not Lie algebra valued, i.e. they hold for the components $\lambda^{a,\alpha,A}$ etc.

e.g. [13]), this can be seen by looking at the fermion susy variation $\delta\lambda \sim F^{\mu\nu}\gamma_{\mu\nu}\epsilon$ (plus terms involving scalars), where ϵ is a four-component Dirac spinor and $\gamma_{\mu\nu}$ are proportional to the generators of the reducible spinorial representation of SO(4) [14]. The latter have the block-diagonal form (A.7), where the matrices $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are anti-selfdual and selfdual respectively, cf. appendix A. Therefore, plugging (2.10) (e.g. for a selfdual field strength F^+) in the susy variation leads (for vanishing scalars) to

$$\delta\lambda \sim \begin{pmatrix} 0 & 0 \\ 0 & (F^+)^{\mu\nu}\bar{\sigma}_{\mu\nu} \end{pmatrix} \begin{pmatrix} \epsilon^+ \\ \epsilon^- \end{pmatrix}, \tag{2.11}$$

showing that the configuration is half-supersymmetric. In the $\mathcal{N} = 1$ theory, the spinors are taken chiral and according to the choice of chirality either the instanton or the anti-instanton represent maximally supersymmetric solutions.

Note that in theories with $\mathcal{N} > 1$, many configurations preserving one or more supersymmetries do not have (anti-)selfdual field strength. We shall see this explicitely below.

A class of finite-action solutions of euclidean Yang-Mills theory was explicitly constructed by Belavin et al. [4] (see also [8, 12] for reviews). The field strength is selfdual and the gauge potential for $k = 1$ and gauge group SU(2) (in regular gauge) takes the form

$$A^a_\mu(x; x_0, \rho) = 2\frac{\eta^a_{\mu\nu}(x - x_0)^\nu}{(x - x_0)^2 + \rho^2}, \tag{2.12}$$

where the arbitrary parameters x_0^μ and ρ are called collective coordinates and $\eta^a_{\mu\nu}$ are the 't Hooft symbols defined in appendix A. Taking into account the gauge orientation, the total number of collective coordinates in this situation is 8 [8]. One can show by computing the index of the Dirac operator in an instanton background (with selfdual field strength) that there are $4Nk$ bosonic collective coordinates for an instanton with winding number k and gauge group SU(N), counting the number of solutions to the selfduality equations with fixed topological charge k [8]. This calculation also reveals that in the background of an (anti-)instanton, the Dirac equation can have non-trivial solutions $\bar{\lambda}_{cl}$ (λ_{cl}) only for negative (positive) chirality spinors, and that the number of such solutions in $2Nk$ [8]. These zero modes are parametrized by the so-called fermionic collective coordinates.

In euclidean super Yang-Mills theories, the previous results can be generalized as follows. First, an obvious solution to the equations of motion is given by (2.10) with all the other fields vanishing. Also, when scalar fields are absent (such as in $\mathcal{N} = 1$ SYM) or uncoupled to fermions, another solution is given by F (anti-)selfdual, $\bar{\lambda} = \bar{\lambda}_{cl}$ ($\lambda = \lambda_{cl}$) and all remaining fields vanishing. But as soon as fermion-scalar couplings are turned on, as in the case of $\mathcal{N} = 4$ SYM, the latter configuration no longer solves the equations of motion (2.6)–(2.9).

For the sake of definiteness, let us remind an iterative way to construct solutions when the gauge group is SU(2) [12]. Start from a configuration $\Phi = (A = A_{cl}, \phi^{AB} = \lambda^A = \bar{\lambda}_A = 0)$, where A_{cl} is a gauge potential for a selfdual field strength. A new solution is obtained by

$$\Phi(\bar{\zeta}) = e^{i\bar{\zeta}_A\bar{Q}^A}\Phi = \sum_{n=0}^{\infty}\frac{1}{n!}\delta^n\Phi, \tag{2.13}$$

where the $\bar{Q}^A$ are the susy generators, and the last equality comes from expanding perturbatively in the fermionic susy parameter $\bar{\zeta}$. Due to the anti-selfduality of $\sigma_{\mu\nu}$, the third equation of (2.5) implies $e^{i\zeta^A Q_A}\Phi = \Phi$, and therefore the positive chirality susy generators cannot be used to generate from Φ a new solution. Using (2.5), and starting from the configuration Φ, one successively obtains

$$
\begin{aligned}
{}^{(0)}A &= A_{cl}\,, \\
{}^{(1)}\bar{\lambda}_A &= -\frac{1}{2}\bar{\sigma}^{\mu\nu}\bar{\zeta}_A{}^{(0)}F_{\mu\nu}\,, \\
{}^{(2)}\phi^{AB} &= \frac{1}{\sqrt{2}}\epsilon^{ABCD}\bar{\zeta}_C{}^{(1)}\bar{\lambda}_D = -\frac{1}{2\sqrt{2}}\epsilon^{ABCD}\bar{\zeta}_C\bar{\sigma}^{\mu\nu}\bar{\zeta}_D{}^{(0)}F_{\mu\nu}\,, \\
{}^{(3)}\lambda^{\alpha,A} &= -\frac{i\sqrt{2}}{3}\bar{\zeta}_{\dot{\alpha},B}\not{D}^{\alpha\dot{\alpha}\,(2)}\phi^{AB} = \frac{i}{6}\epsilon^{ABCD}\sigma_\rho{}^{\alpha\dot{\beta}}\bar{\zeta}_{\dot{\beta},B}\left(\bar{\zeta}_C\bar{\sigma}^{\mu\nu}\bar{\zeta}_D\right)D^{\rho\,(0)}F_{\mu\nu}\,, \\
{}^{(4)}A_\mu &= -\frac{i}{4}\bar{\zeta}_A\bar{\sigma}_\mu{}^{(3)}\lambda^A = \frac{1}{24}\epsilon^{ABCD}\left(\bar{\zeta}_A\bar{\sigma}_{\mu\rho}\bar{\zeta}_B\right)\left(\bar{\zeta}_C\bar{\sigma}^{\sigma\nu}\bar{\zeta}_D\right)D^{\rho\,(0)}F_{\sigma\nu}\,, \\
&\vdots
\end{aligned}
\tag{2.14}
$$

where the superscript indicates the number of susy parameters $\bar{\zeta}$ contained in the field. It turns out that $\bar{\zeta}$ is one of the two two-component fermionic collective coordinates for the SU(2) gauge group. The other one is obtained by using superconformal supersymmetry transformation laws [12]. The fact that all fermionic zero modes can be generated by means of ordinary supersymmetry and superconformal transformations is specific to SU(2), and the situation is more involved for SU($N > 2$) [12]. The solution constructed iteratively in this way is called *super-instanton* (cf. [5, 12, 13] for reviews).

3 Spinorial geometry of four-dimensional euclidean space

In this section we summarize the essential information needed to realize spinors of Spin(4) in terms of forms [6]. For more details, we refer to [15]. Consider the real vector space $V = \mathbb{R}^4$ endowed with its canonical scalar product and orthonormal basis $\{e_1, e_2, e_3, e_4\}$. Define the subspace U spanned by the first two basis elements, $U = \mathrm{span}(e_1, e_2)$. The space of Dirac spinors $\Delta_{\mathbb{C}}$ is defined as the exterior algebra of $U \otimes \mathbb{C}$,

$$\Delta_{\mathbb{C}} = \Lambda^*(U \otimes \mathbb{C})\,, \tag{3.1}$$

which is nothing else than $\mathrm{Span}_{\mathbb{C}}(1, e_1, e_2, e_1 \wedge e_2 =: e_{12})$. A generic Dirac spinor is thus written as

$$\lambda = \lambda_0 1 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_{12} e_{12}\,, \tag{3.2}$$

and has 4 complex degrees of freedom as it should. The gamma matrices are represented on $\Delta_{\mathbb{C}}$ as

$$
\begin{aligned}
\gamma_i\lambda &= e_i \wedge \lambda + e_i \lrcorner \lambda\,, \\
\gamma_{i+2}\lambda &= ie_i \wedge \lambda - ie_i \lrcorner \lambda\,,
\end{aligned}
\tag{3.3}
$$

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where $i = 1, 2$, and the contraction operator $\lrcorner$ is defined though its action on a k-form as

$$e_i \lrcorner \left(\frac{1}{k!} \eta_{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k} \right) = \frac{1}{(k-1)!} \eta_{i j_1 \ldots j_{k-1}} e_{j_1} \wedge \ldots \wedge e_{j_{k-1}} \ . \tag{3.4}$$

One easily checks that this representation of the gamma matrices satisfies the Clifford algebra relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Note that the elements e_i are indistinctly viewed as vectors or forms according to the objects on which they are acting. From the definition of the chirality matrix $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ and (3.3), one readily sees that

$$\begin{aligned} \gamma_5 1 &= 1 \,, & \gamma_5 e_{12} &= e_{12} \,, \\ \gamma_5 e_1 &= -e_1 \,, & \gamma_5 e_2 &= -e_2 \,, \end{aligned} \tag{3.5}$$

so that the usual split of the space of Dirac spinors into positive and negative chirality Weyl spinors here amounts to the split $\Delta_{\mathbb{C}} = \Delta^+ \oplus \Delta^-$ into forms of even degree $1, e_{12}$ and forms of odd degree e_1, e_2. This decomposition is invariant under the euclidean Lorentz group $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, with each subspace transforming under a different SU(2) factor.

Let us define the hermitian inner product

$$\langle a^i e_i | b^j e_j \rangle = \sum_{i=1}^{2} a^{*i} b^i \tag{3.6}$$

on $U \otimes \mathbb{C}$, and then extend it to $\Delta_{\mathbb{C}}$. This yields the Spin(4) invariant Dirac inner product on the space of spinors $\Delta_{\mathbb{C}}$,

$$D(\eta, \theta) = \langle \eta | \theta \rangle \ . \tag{3.7}$$

It reveals quite useful for practical purposes to switch to another basis for the gamma matrices, defining

$$\begin{aligned} \Gamma_1 &= \frac{1}{\sqrt{2}} (\gamma_1 - i\gamma_3) \ , & \Gamma_{\bar{1}} &= \frac{1}{\sqrt{2}} (\gamma_1 + i\gamma_3) \ , \\ \Gamma_2 &= \frac{1}{\sqrt{2}} (\gamma_2 - i\gamma_4) \ , & \Gamma_{\bar{2}} &= \frac{1}{\sqrt{2}} (\gamma_2 + i\gamma_4) \ . \end{aligned} \tag{3.8}$$

In this new basis, the gamma matrices satisfy $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$, $A, B = \{1, \bar{1}, 2, \bar{2}\}$, where the non-vanishing components of the metric are $\eta_{1\bar{1}} = \eta_{\bar{1}1} = 1$, $\eta_{2\bar{2}} = \eta_{\bar{2}2} = 1$.

The advantage of this new basis stems from the fact that the Γ_A satisfy a fermionic annihilation-creation operator algebra, since $\{\Gamma_1, \Gamma_{\bar{1}}\} = 2$, $\{\Gamma_2, \Gamma_{\bar{2}}\} = 2$, for which the spinor 1 can be identified as the vacuum state, being annihilated by $\Gamma_{\bar{1}}$ and $\Gamma_{\bar{2}}$:

$$\Gamma_{\bar{1}} 1 = \Gamma_{\bar{2}} 1 = 0 \ . \tag{3.9}$$

All the other states can be constructed by acting with Γ_1 and Γ_2 on 1. Using (3.8) and (3.3), one can compute the action of the gamma matrices and the Spin(4) generators $\Gamma_{AB} = \Gamma_{[A}\Gamma_{B]}$ on the basis spinors. This is summarized in table 1, where another simplification coming from the use of the basis (3.8) is apparent from the vanishing of half of the entries of this table.[3]

[3]Spinors that are annihilated by half of the gamma matrices are sometimes referred to as *pure spinors*. Table 1 is a manifestation of the fact that in 6 or less real dimensions, all spinors are pure.

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	1	e_1	e_2	e_{12}
Γ_1	$\sqrt{2}e_1$	0	$\sqrt{2}e_{12}$	0
$\Gamma_{\bar{1}}$	0	$\sqrt{2}$	0	$\sqrt{2}e_2$
Γ_2	$\sqrt{2}e_2$	$-\sqrt{2}e_{12}$	0	0
$\Gamma_{\bar{2}}$	0	0	$\sqrt{2}$	$-\sqrt{2}e_1$
$\Gamma_{1\bar{1}}$	-1	e_1	$-e_2$	e_{12}
Γ_{12}	$2e_{12}$	0	0	0
$\Gamma_{1\bar{2}}$	0	0	$2e_1$	0
$\Gamma_{\bar{1}2}$	0	$-2e_2$	0	0
$\Gamma_{\bar{1}\bar{2}}$	0	0	0	-21
$\Gamma_{2\bar{2}}$	-1	$-e_1$	e_2	e_{12}

Table 1. Action of the gamma matrices and Spin(4) generators on the basis $1, e_1, e_2, e_{12}$.

We will sometimes use another basis, in which the gamma matrices are given by (A.5). The spinors $\{1, e_1, e_2, e_{12}\}$ can easily be expressed in that basis by starting from (3.9) and acting with the creation operators on the vacuum. Their form is fixed up to a global phase by normalizing the states to unit norm:

$$1^\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad e_{12}^\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix},$$
$$e_{1\dot\alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad e_{2\dot\alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ -1 \end{pmatrix}, \tag{3.10}$$

where we suppressed two lower zeroes in $1, e_{12}$ and two upper zeroes in e_1, e_2. (3.10) provides thus an expression of the forms in usual two-component notation. For practical purposes, we also mention their complex conjugates,

$$(1^\alpha)^* = -ie_{12}^\alpha\,, \qquad (e_{12}^\alpha)^* = -i1^\alpha\,, \qquad (e_{1\dot\alpha})^* = ie_{2\dot\alpha}\,, \qquad (e_{2\dot\alpha})^* = ie_{1\dot\alpha}\,, \tag{3.11}$$

the corresponding spinors with raised/lowered indices,

$$1_\alpha = 1^\beta \epsilon_{\beta\alpha} = i1^\alpha\,, \qquad e_{12\alpha} = e_{12}^\beta \epsilon_{\beta\alpha} = -ie_{12}^\alpha\,,$$
$$e_1^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta} e_{1\dot\beta} = -ie_{1\dot\alpha}\,, \qquad e_2^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta} e_{2\dot\beta} = ie_{2\dot\alpha}\,,$$

as well as the various contractions of the basis spinors,

$$1^\alpha e_{12\alpha} = 1\,, \qquad e_{12}^\alpha 1_\alpha = -1\,, \qquad e_1^{\dot\alpha} e_{2\dot\alpha} = -1\,, \qquad e_2^{\dot\alpha} e_{1\dot\alpha} = 1\,. \tag{3.12}$$

Finally, let us express for further reference the (anti-)selfduality conditions (2.10) in terms of the basis (3.8):

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \qquad \Leftrightarrow \qquad F^{1\bar{1}} + F^{2\bar{2}} = 0\,, \qquad F^{12} = 0\,, \tag{3.13}$$

$$F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \qquad \Leftrightarrow \qquad F^{1\bar{1}} - F^{2\bar{2}} = 0\,, \qquad F^{1\bar{2}} = 0\,. \tag{3.14}$$

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In these coordinates, the complex conjugation simply amounts to change a barred index into an unbarred one, e.g. $(V^{\bar{i}})^* = V^i$, $(F^{i\bar{j}})^* = F^{\bar{i}j}$, etc. Therefore $F^{1\bar{1}}$ and $F^{2\bar{2}}$ are purely imaginary, and the equations (3.13), (3.14) each impose three real conditions on the components of $F^{\mu\nu}$ as they should. (3.13) and (3.14) are sometimes referred to as the *Donaldson equations*. We will encounter in the next section generalizations thereof including in particular the scalar fields present in $\mathcal{N} = 4$ SYM.

4 Classification of supersymmetric solutions

4.1 Representatives

The spinorial geometry approach is tailored to fully exploit the linearity of the Killing spinor equations. One of its basic ingredients is the use of the symmetries of the theory to transform the Killing spinors to preferred representatives of their orbit under this symmetry group. This is the scope of the present subsection.

Using the results of the previous section, the Killing spinors appearing in (2.5) can be expressed in the language of forms as

$$\begin{aligned} \xi^{\alpha,A} &= \omega_0^A 1^\alpha + \omega_{12}^A e_{12}^\alpha \,, \\ \bar{\xi}_{\dot{\alpha},A} &= \omega_{1,A} e_{1\dot{\alpha}} + \omega_{2,A} e_{2\dot{\alpha}} \,, \end{aligned} \tag{4.1}$$

where the coefficients ω^A $(A = 1, \ldots, 4)$ are complex numbers that are related by the symplectic Majorana condition (2.2), which imposes the following structure:

$$\omega_0 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} , \qquad \omega_{12} = \begin{pmatrix} -d^* \\ -c^* \\ b^* \\ a^* \end{pmatrix} , \qquad \omega_1 = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} , \qquad \omega_2 = \begin{pmatrix} h^* \\ g^* \\ -f^* \\ -e^* \end{pmatrix} . \tag{4.2}$$

This structure is preserved both by the SO(4) Lorentz and by the SO(5,1) internal R-symmetry transformations [8]. We are now going to use the latter symmetries to simplify the form (4.1)–(4.2) of a generic Killing spinor. These are generated by the $\gamma_{\mu\nu}$ and $\hat{\gamma}_{ab}$ given respectively in (A.7) and (A.13).

First, we notice that for a right chiral spinor there exists a single orbit for the action of SO(5,1), and consequently a single representative that can be brought to the form

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e_{12} \,. \tag{4.3}$$

To see this, it is enough to find a series of transformations bringing the vector $(1, 0, 0, 0)^T$ to $(a, b, c, d)^T$, for arbitrary $a, b, c, d \in \mathbb{C}$. This is done for instance by

$$e^{\alpha\Sigma^{25}} e^{\beta\Sigma^{14}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} e^{\beta} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{4.4}$$

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with $a_1 \in \mathbb{C}$, followed by

$$e^{\alpha\Sigma^{25}} e^{\beta\Sigma^{45}} \begin{pmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \cos\beta e^{-i\alpha} \\ -a_1 \sin\beta e^{i\alpha} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ 0 \\ 0 \end{pmatrix} , \tag{4.5}$$

where $a_2, b_2 \in \mathbb{C}$ are independent, and finally

$$e^{\frac{\alpha}{2}[\Sigma^{25}-\Sigma^{36}]} e^{\frac{\beta}{2}[\Sigma^{23}+\Sigma^{56}]} e^{\frac{\gamma}{2}[\Sigma^{25}+\Sigma^{36}]} e^{\frac{\delta}{2}[\Sigma^{23}-\Sigma^{56}]} \begin{pmatrix} a_2 \\ b_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_2 \cos\delta e^{-i\gamma} \\ b_2 \cos\beta e^{i\alpha} \\ -b_2 \sin\beta e^{-i\alpha} \\ -a_2 \sin\delta e^{i\gamma} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} .$$

Similarly, there exists a single orbit, and hence a single representative under SO(5, 1) for left chiral spinors.

After having brought a generic chiral Killing spinor to a simpler form ((4.3) for a right spinor), one looks for the subgroup of the global symmetry group that leaves it invariant. This stability subgroup can then be used to simplify additional Killing spinors. For the representative (4.3), one checks that the SO(5, 1) generators stabilizing it are

$$\begin{aligned} &S_1 = \frac{1}{2}\left(\hat\gamma^{25} - \hat\gamma^{36}\right) , \quad S_2 = \frac{1}{2}\left(\hat\gamma^{23} + \hat\gamma^{56}\right) , \quad S_3 = \frac{1}{2}\left(\hat\gamma^{26} + \hat\gamma^{35}\right) , \\ &S_4 = \frac{1}{2}\left(\hat\gamma^{15} + \hat\gamma^{45}\right) , \quad S_5 = \frac{1}{2}\left(\hat\gamma^{12} - \hat\gamma^{24}\right) , \quad S_6 = \frac{1}{2}\left(\hat\gamma^{16} + \hat\gamma^{46}\right) , \quad S_7 = \frac{1}{2}\left(\hat\gamma^{13} - \hat\gamma^{34}\right) , \end{aligned} \tag{4.6}$$

where S_1, S_2 and S_3 form an $su(2)$ subalgebra, while $S_4, \cdots, S_7$ form an abelian ideal. Being a right chiral spinor of Spin(4), it is also stabilized by the $\mathrm{SU}(2)_L$ subgroup of Spin(4) generated by $\Gamma_{1\bar1} - \Gamma_{2\bar2}$, $\Gamma_{1\bar2}$ and $\Gamma_{\bar12}$. On the other hand, the $\mathrm{SU}(2)_R$ generators $\Gamma_{1\bar1} + \Gamma_{2\bar2}$, Γ_{12} and $\Gamma_{\bar1\bar2}$ only mix the forms 1 and e_{12} (Spin(4) transformations preserve chirality), so that applying an $\mathrm{SU}(2)_R$ transformation to the representative (4.3) yields again a spinor of the form $\omega_0 1 + \omega_{12} e_{12}$, with ω_0, ω_{12} given in (4.2). But, as we just explained, that spinor is in the same orbit as (4.3) under SO(5, 1). The action of the $\mathrm{SU}(2)_R$ can thus be compensated by a subsequent SO(5, 1) transformation. It is easy to see that the SO(5, 1) generators accomplishing this are

$$S_8 = \frac{1}{2}(\hat\gamma^{25} + \hat\gamma^{36}) , \qquad S_9 = \frac{1}{2}(\hat\gamma^{23} - \hat\gamma^{56}) , \qquad S_{10} = \frac{1}{2}(\hat\gamma^{26} - \hat\gamma^{35}) , \tag{4.7}$$

which of course span an $su(2)$ algebra that commutes with the one formed by $S_1, \ldots, S_3$. Together, $S_1, \ldots, S_{10}$ generate the Euclidean group $\mathrm{SO}(4) \ltimes \mathbb{R}^4 \cong ISO(4)$, so that the stability subgroup of (4.3) is $\mathrm{SU}(2)_L \times ISO(4)$.

Let us see how a second Killing spinor can be simplified using the stability subgroup of the first. It can have either the same chirality, or the opposite one.[4] First assume that

[4]Here, we will always consider Killing spinors with a definite chirality. This is restrictive, but as far as instantons are concerned, it is the most important case.

the second one has the same chirality. It can thus be written as

$$\xi_2 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} 1 + \begin{pmatrix} -d^* \\ -c^* \\ b^* \\ a^* \end{pmatrix} e_{12} \, . \tag{4.8}$$

One observes that when the second spinor is of the form

$$\xi_2^{(1)} = \begin{pmatrix} a \\ 0 \\ 0 \\ d \end{pmatrix} 1 + \begin{pmatrix} -d^* \\ 0 \\ 0 \\ a^* \end{pmatrix} e_{12} \, , \tag{4.9}$$

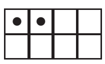
we may use an $\mathrm{SU}(2)_R$ transformation (compensated by $S_8, \dots, S_{10}$) to choose $a, d \in \mathbb{R}$. The residual isotropy group leaving invariant (4.3) as well as (4.9) is then given by $\mathrm{SU}(2)_L \times ((\mathrm{U}(1) \times \mathrm{SU}(2)) \ltimes \mathbb{R}^4)$, with the $\mathrm{U}(1)$ generated by S_9. (4.9) actually belongs to the same Lorentz orbit as (4.3), in that an arbitrary right $\mathrm{SO}(4)$ chiral spinor $a1 - d^* e_{12}$ can always be brought to the spinor $\rho^2 1$, $\rho \in \mathbb{R}$ by means of an $\mathrm{SO}(4)$ transformation (to show this, consider e.g. $\exp(\alpha\gamma^{13})\exp(\beta\gamma^{23})\exp(\delta\gamma^{13})$ acting on $\rho^2 1$). Actually, since the Killing spinor equations (2.5) are linear in the spinorial parameter, we will generally forget about the scaling factor ρ^2.

On the other hand, when b or c are different from zero, the second spinor can, up to an overall factor, be brought to the form

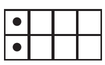
$$\xi_2^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e_{12} \, , \tag{4.10}$$

using the generators $S_1, \dots, S_7$ (this is obvious by acting on (4.10) with $e^{\alpha S_1} e^{\beta S_2} e^{\gamma S_1} e^{\delta S_7} e^{\eta S_6} e^{\theta S_5} e^{\Omega S_4}$). The spinor (4.10) is further stabilized by S_8, S_9 and S_{10}.

We will use the following pictorial representation. Without loss of generality, the first chiral Killing spinor can always be taken of the form (4.3). Next, a configuration admitting as Killing spinors (4.3) and (4.9) will be represented by

,

while a configuration admitting as Killing spinors (4.3) and (4.10) shall be denoted by

.

Each dot represents a Killing spinor. A solution can have at most 8 real Killing spinors of the same chirality, here right. Spinors that can be related by a Spin(4) transformation are written on the same line. In the second case, for example, the two Killing spinors can only be related by an internal $\mathrm{SO}(5,1)$ transformation.

One can also consider Killing spinors of different chiralities. Let us again fix the first spinor to be of the form (4.3), and use its stability subgroup to simplify a left chiral spinor

$$\bar{\xi} = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} e_1 + \begin{pmatrix} h^* \\ g^* \\ -f^* \\ -e^* \end{pmatrix} e_2 \,. \tag{4.11}$$

There are again two cases. If e or h is different from zero, one can use the Lorentz subgroup $\mathrm{SU}(2)_L$ as well as $S_4, \ldots, S_7$ to bring the left chiral spinor to

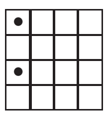
$$\bar{\xi}_2^{(1)} = \begin{pmatrix} \rho^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} e_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\rho^2 \end{pmatrix} e_2 \,. \tag{4.12}$$

This spinor is still invariant under SO(4) transformations generated by $S_1, \ldots, S_3$ and $S_8, \ldots, S_{10}$. On the other hand, if $e = h = 0$, the generators of the $\mathrm{SU}(2)_L$ subgroup (or equivalently the generators S_1, S_2 and S_3) can be used to cast it into the form

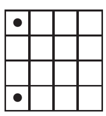
$$\bar{\xi}_2^{(2)} = \begin{pmatrix} 0 \\ \rho^2 \\ 0 \\ 0 \end{pmatrix} e_1 + \begin{pmatrix} 0 \\ 0 \\ -\rho^2 \\ 0 \end{pmatrix} e_2 \,, \tag{4.13}$$

which is still stabilized by $S_4, \ldots, S_{10}$. Note that (4.12) and (4.13) are actually related by $\mathrm{SO}(5,1)$ transformations, but not of the type (4.6) stabilizing (4.3). This actually shows that a generic Dirac spinor can be brought either to the form (4.3)+(4.12) or to (4.3)+(4.13), i.e. there are two orbits under the Lorentz and internal symmetry groups for a generic Killing spinor.

Allowing for Killing spinors of different chiralities, a solution can preserve at most 16 real supersymmetries. Again, we will make use of a pictorial representation to visualize the different possible supersymmetric configurations in an easy way. A solution admitting (4.3) and (4.12) as Killings spinors will be written as

 $\equiv (1,0,1,0)\,,$

while one admitting (4.3) and (4.13) is denoted by

 $\equiv (1,0,0,1)\,.$

We make use of the shorthand notation (n_1, n_2, n_3, n_4) to indicate the number n_i of Killing spinors lying in the i^{th} Spin(4) orbit, whose representatives are given respectively

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by (4.3), (4.10), (4.12), (4.13). There are obviously symmetries of these diagrams that would yield equivalent configurations. For example, up to the choice of chirality of the first Killing spinor, the configurations obtained by exchanging the first two lines with the last two ones are equivalent, e.g. $(2,0,1,0) = (1,0,2,0)$, $(4,0,0,0) = (0,0,4,0)$ etc.

4.2 Bosonic configurations

Let us now use the machinery of the previous section to classify the purely bosonic configurations. We therefore put all fermions to zero in (2.5), and are thus left with

$$\begin{aligned}
\delta\lambda^{\alpha,A} &= -\tfrac{1}{2}\sigma^{\mu\nu\alpha}{}_{\beta}F_{\mu\nu}\xi^{\beta,A} - i\sqrt{2}\bar{\xi}_{\dot{\alpha},B}\not{D}^{\alpha\dot{\alpha}}\phi^{AB} + \left[\phi^{AB},\bar{\phi}_{BC}\right]\xi^{\alpha,C} = 0\,,\\
\delta\bar{\lambda}_{\dot{\alpha},A} &= -\tfrac{1}{2}\bar{\sigma}^{\mu\nu}{}_{\dot{\alpha}}{}^{\dot{\beta}}F_{\mu\nu}\bar{\xi}_{\dot{\beta},A} + i\sqrt{2}\xi^{\alpha,B}\not{D}_{\dot{\alpha}\alpha}\bar{\phi}_{AB} + \left[\bar{\phi}_{AB},\phi^{BC}\right]\bar{\xi}_{\dot{\alpha},C} = 0\,.
\end{aligned}\tag{4.14}$$

These equations can be rewritten to emphasize the action of the operators in table 1 and avoiding explicit spinor indices:

$$\begin{aligned}
-\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\xi^{A} + \sqrt{2}\Gamma_{\mu}\bar{\xi}_{B}D^{\mu}\phi^{AB} + \left[\phi^{AB},\bar{\phi}_{BC}\right]\xi^{C} &= 0\,,\\
-\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\bar{\xi}_{A} + \sqrt{2}\Gamma_{\mu}\xi^{B}D^{\mu}\bar{\phi}_{AB} + \left[\bar{\phi}_{AB},\phi^{BC}\right]\bar{\xi}_{C} &= 0\,.
\end{aligned}\tag{4.15}$$

Plugging (4.1) into these equations, using table 1, and requiring the coefficients of 1, e_1, e_2 and e_{12} to vanish yields the following system of 16 complex equations:

$$\begin{aligned}
\omega_0^A(F^{1\bar{1}}+F^{2\bar{2}})+2\omega_{12}^A F^{\bar{1}\bar{2}}+2D^{\bar{1}}\phi^{AB}\omega_{1,B}+2D^{\bar{2}}\phi^{AB}\omega_{2,B}+\left[\phi^{AB},\bar{\phi}_{BC}\right]\omega_0^C &= 0\,,\\
-\omega_{12}^A(F^{1\bar{1}}+F^{2\bar{2}})-2\omega_0^A F^{12}+2D^{1}\phi^{AB}\omega_{2,B}-2D^{2}\phi^{AB}\omega_{1,B}+\left[\phi^{AB},\bar{\phi}_{BC}\right]\omega_{12}^C &= 0\,,\\
-\omega_{1,A}(F^{1\bar{1}}-F^{2\bar{2}})-2\omega_{2,A}F^{1\bar{2}}+2D^{1}\bar{\phi}_{AB}\omega_0^B-2D^{\bar{2}}\bar{\phi}_{AB}\omega_{12}^B+\left[\bar{\phi}_{AB},\phi^{BC}\right]\omega_{1,C} &= 0\,,\\
\omega_{2,A}(F^{1\bar{1}}-F^{2\bar{2}})+2\omega_{1,A}F^{\bar{1}2}+2D^{\bar{1}}\bar{\phi}_{AB}\omega_{12}^B+2D^{2}\bar{\phi}_{AB}\omega_0^B+\left[\bar{\phi}_{AB},\phi^{BC}\right]\omega_{2,C} &= 0\,.
\end{aligned}\tag{4.16}$$

However, due to the conditions (4.2), the equations in the first and third line of (4.16) are related to those in the second and fourth line by complex conjugation. We are thus left with 16 independent real equations.

Without loss of generality, one can choose the first Killing spinor to be (4.3), for which the system (4.16) boils down to

$$\begin{aligned}
F^{1\bar{1}} + F^{2\bar{2}} - [\phi^{12},\phi^{34}] + [\phi^{13},\phi^{24}] &= 0\,,\\
F^{\bar{1}\bar{2}} + [\phi^{24},\phi^{34}] &= 0\,,\\
D^{1}\phi^{34} - D^{\bar{2}}\phi^{13} &= 0\,,\\
D^{1}\phi^{24} - D^{\bar{2}}\phi^{12} &= 0\,,
\end{aligned}\tag{4.17}$$

as well as

$$[\phi^{23},\phi^{AB}] = D^{\mu}\phi^{23} = 0\,.\tag{4.18}$$

If we define a Lie-algebra valued one-form ϕ (the "Higgs field") with components

$$\phi_1 = (\phi_{\bar{1}})^* = \phi^{34}\,,\qquad \phi_2 = (\phi_{\bar{2}})^* = \phi^{24}\,,\tag{4.19}$$

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the system (4.17) can be rewritten in the simple form

$$(F-\phi\wedge\phi)^{1\bar{1}}+(F-\phi\wedge\phi)^{2\bar{2}}=0\,,\qquad (F-\phi\wedge\phi)^{12}=0\,,\tag{4.20}$$

$$(D\phi)_{1\bar{1}}-(D\phi)_{2\bar{2}}=0\,,\qquad (D\phi)_{1\bar{2}}=0\,,\tag{4.21}$$

together with

$$D\star\phi=0\,,\tag{4.22}$$

where $\star\phi$ denotes the Hodge dual of ϕ. (4.20) means that the combination $F-\phi\wedge\phi$ must be selfdual, while (4.21) is nothing else than the anti-selfduality condition for $D\phi$. Similar equations appeared previously in [16, 17].[5] The reason of why four of the scalars combine to a one-form (which at first sight appears to transform differently under Lorentz transformations) lies in the stability subgroup of the spinor (4.3): as was explained in section 4.1, once we fix the representative (4.3), we are free to do $\mathrm{SO}(4)\cong\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ Lorentz rotations only if the $\mathrm{SU}(2)_R$ is compensated by a subsequent $\mathrm{SO}(5,1)$ transformation, and the scalar fields do transform under the latter. A similar situation occurs in twisted theories.

Using the complex-valued connection

$$\mathcal{A}=A+i\phi\,,\tag{4.23}$$

(4.20) and (4.21) are equivalent to

$$\mathcal{F}=\star\bar{\mathcal{F}}\,,\tag{4.24}$$

where $\mathcal{F}$ is the field strength of $\mathcal{A}$ and $\bar{\mathcal{F}}$ denotes its complex conjugate. (4.20)–(4.22) bear some resemblance to the Hitchin equations [18]

$$\begin{aligned}F-\phi\wedge\phi&=0\,,\\ D\phi=D\star\phi&=0\,.\end{aligned}\tag{4.25}$$

Note however that in (4.25), A is a connection on a G-bundle $E\to C$, with C a Riemann surface, while in our context, A is a connection on a bundle over four-dimensional euclidean space. Moreover, (4.25) imply that $\mathcal{A}=A+i\phi$ is flat, whereas here $\mathcal{F}$ satisfies the selfduality-like condition (4.24). Notice also that the Hitchin equations arise by reduction of the selfduality equations from four to two dimensions [18].[6] This raises the question of whether the system (4.20)–(4.22) also has a higher-dimensional origin. This is indeed the case: Consider the higher-dimensional analogue of the selfduality equations [20],

$$\frac{1}{2}T_{\mu\nu\rho\sigma}F^{\rho\sigma}=\lambda F_{\mu\nu}\,,\tag{4.26}$$

[5] Ref. [17] deals with a twisted version of $\mathcal{N}=4$ SYM that is relevant to the geometric Langlands program. This gives a family of topological field theories parametrized by some t that takes values in the one-dimensional complex projective space. Taking $t\to\infty$ in eqs. (3.29) of [17] yields our system (4.20)–(4.22).

[6] Reduction of the selfduality equations from four to three dimensions yields the monopole equations. In gravity, the four-dimensional self-duality equations with one Killing direction imply the 3d Einstein-Weyl equations [19].

where λ is a number and the tensor $T_{\mu\nu\rho\sigma}$ is totally antisymmetric. If the dimension D is higher than four, $T_{\mu\nu\rho\sigma}$ cannot be invariant under $\mathrm{SO}(D)$ anymore. The authors of [20] classified all possible choices for $T_{\mu\nu\rho\sigma}$ up to $D=8$, requiring that $T_{\mu\nu\rho\sigma}$ be invariant under a maximal subgroup of $\mathrm{SO}(D)$. They found that the case $D=8$ is of particular interest, because it generalizes most closely the concept of four-dimensional selfduality. For $D=8$ and the choice Spin(7) as maximal subgroup of SO(8) there are two possible eigenvalues $\lambda=1$ and $\lambda=-3$ [20]. The former leads to the set of seven equations[7]

$$\begin{aligned}
F_{32}+F_{14}+F_{56}+F_{78} &= 0\,,\\
F_{31}+F_{42}+F_{57}+F_{86} &= 0\,,\\
F_{34}+F_{21}+F_{76}+F_{85} &= 0\,,\\
F_{35}+F_{62}+F_{71}+F_{48} &= 0\,,\\
F_{36}+F_{25}+F_{18}+F_{47} &= 0\,,\\
F_{37}+F_{82}+F_{15}+F_{64} &= 0\,,\\
F_{38}+F_{27}+F_{61}+F_{54} &= 0\,,
\end{aligned} \tag{4.27}$$

called octonionic instanton equations, since they can be rephrased using the structure constants of the octonions [20]. Now decompose the vector potential as

$$A_M = (A_\mu, -\phi_3, \phi_2, \phi_1, -\phi_4)\,,$$

where $M=1,\ldots,8$ and $\mu=1,\ldots,4$, and suppose that the fields A_μ, ϕ_μ are independent of the coordinates $x^5,\ldots,x^8$. Then the octonionic instanton equations (4.27) yield exactly the system (4.20)–(4.22).[8]

Looking at (4.18), we observe that the field ϕ^{23} is covariantly constant and commutes with any other scalar. In what follows, we will refer to these conditions as *decoupling conditions*. Moreover, the field ϕ^{14} does not appear in the susy equations (except through the fact that it commutes with ϕ^{23}). This implies that the ghost $i(\phi^{14}+\phi^{23})$ decouples from the other fields in the supersymmetry constraints.

Comparing (4.17) to (3.13), we see that in presence of scalars the field strength F is no longer selfdual, but the complex field strength $\mathcal{F}$ does obey the selfduality-like equation (4.24). It would be interesting to see if one can use (4.24) to construct generalizations of instantons to include nonvanishing scalars.[9]

We shall now analyze what happens when requiring the existence of more Killing spinors.

4.2.1 Killing spinors of same chirality

One can add a further Killing spinor of the same chirality to the original configuration $(1,0,0,0)$ in two different ways, either one in the same Lorentz orbit as the first, namely $(2,0,0,0)$, or the other one $(1,1,0,0)$.

[7] With respect to [20], we interchanged the 1- and 3-directions.

[8] The relation of BPS equations in euclidean $\mathcal{N}=4$ SYM to the octonionic instanton equations was noticed before in [16] for the case of two active scalars.

[9] A particular type of such solutions, termed ic-instantons, was constructed explicitely in [16].

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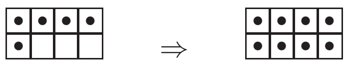
Let us first consider the former case, i.e., we take a second spinor of the form (4.9). Equations (4.17) are supplemented with

$$
\begin{aligned}
\mathrm{Im}\left(d\left[\phi^{12},\phi^{13}\right]\right) &= 0\,,\\
d\left(\left[\phi^{12},\phi^{34}\right]-\left[\phi^{13},\phi^{24}\right]\right)+2i\mathrm{Im}(a)\left[\phi^{24},\phi^{34}\right] &= 0\,,\\
-2i\mathrm{Im}(a)D^{\bar 2}\phi^{13}-d^{*}D^{\bar 2}\phi^{34}-dD^{1}\phi^{13} &= 0\,,\\
2i\mathrm{Im}(a)D^{\bar 2}\phi^{12}+d^{*}D^{\bar 2}\phi^{24}+dD^{1}\phi^{12} &= 0\,.
\end{aligned}
\tag{4.28}
$$

From these equations, one easily gets the constraints coming from the presence of more Killing spinors on the same Lorentz orbit, i.e. configurations $(3,0,0,0)$ and $(4,0,0,0)$. First, one may verify that the constraints coming by considering an additional third Killing spinor forces the field strength to be selfdual, while giving further constraints on the covariant derivatives and commutators of the scalar fields. For $(4,0,0,0)$, one may combine (4.17) with the equations obtained from (4.28) with $(a,d)=(i,0),(0,1),(0,i)$ which generate a basis for the entire orbit. We obtain the following conditions:

$$
(4,0,0,0)\qquad\Leftrightarrow\qquad
\left\{
\begin{array}{rll}
F^{1\bar 1}+F^{2\bar 2} = 0\,, & & F^{12}=0\,,\\
\left[\phi^{23},\phi^{AB}\right] = 0 & & \forall\ A,B\,,\\
D^{\mu}\phi^{AB} = 0 & & \forall\ A,B \text{ but for } \phi^{14}\,,\\
\left[\phi^{24},\phi^{34}\right]=\left[\phi^{12},\phi^{13}\right] = 0\,, & & \\
\left[\phi^{12},\phi^{34}\right] = \left[\phi^{13},\phi^{24}\right]. & &
\end{array}
\right.
\tag{4.29}
$$

The scalar field ϕ^{14} is thus the only field which is left completely unconstrained (except from its vanishing commutator with ϕ^{23}). Adding one more chiral Killing spinor to $(4,0,0,0)$ directly leads to the instanton solution with selfdual field strength, and vanishing covariant derivatives and commutators for all scalar fields. It is clear from (4.14) that this solution preserves 8 real supersymmetries, thus



We now focus on the latter case, i.e. $(1,1,0,0)$. Combining (4.17) with the constraints coming from (4.10), one gets

$$
\begin{aligned}
F^{1\bar 1}+F^{2\bar 2}-\left[\phi^{12},\phi^{34}\right] &= 0\,,\\
F^{\bar 1\bar 2}+\left[\phi^{24},\phi^{34}\right] &= 0\,,\\
D^{1}\phi^{34}-D^{\bar 2}\phi^{13} &= 0\,,\\
D^{1}\phi^{24}-D^{\bar 2}\phi^{12} &= 0\,,\\
\left[\phi^{23},\phi^{AB}\right]=\left[\phi^{14},\phi^{AB}\right]=\left[\phi^{13}+\phi^{24},\phi^{AB}\right] &= 0\,,\\
D^{\mu}\phi^{23}=D^{\mu}\phi^{14}=D^{\mu}(\phi^{13}+\phi^{24}) &= 0\,.
\end{aligned}
\tag{4.30}
$$

Therefore three purely imaginary fields decouple. Let us consider the constraints coming from the existence of a third spinor of the same chirality. The latter is completely generic and is of the form (4.1)–(4.2). Taking into account (4.30), the system one obtains is

$$\begin{aligned}
\mathrm{Im}(d\left[\phi^{12},\phi^{24}\right]) &= 0\,,\\
\mathrm{Im}(c\left[\phi^{12},\phi^{24}\right]) &= 0\,,\\
2i\mathrm{Im}(b)\left[\phi^{24},\phi^{34}\right] + c\left[\phi^{12},\phi^{34}\right] &= 0\,,\\
2i\mathrm{Im}(a)\left[\phi^{24},\phi^{34}\right] + d\left[\phi^{12},\phi^{34}\right] &= 0\,,\\
2i\mathrm{Im}(b)D^{\bar{2}}\phi^{24} + c^* D^{\bar{2}}\phi^{34} - cD^{1}\phi^{24} &= 0\,,\\
2i\mathrm{Im}(a)D^{\bar{2}}\phi^{24} + d^* D^{\bar{2}}\phi^{34} - dD^{1}\phi^{24} &= 0\,,\\
2i\mathrm{Im}(b)D^{1}\phi^{24} + cD^{1}\phi^{12} + c^* D^{\bar{2}}\phi^{24} &= 0\,,\\
2i\mathrm{Im}(a)D^{1}\phi^{24} + dD^{1}\phi^{12} + d^* D^{\bar{2}}\phi^{24} &= 0\,.
\end{aligned} \tag{4.31}$$

The precise form of the equations is not really important, but what is worth noticing is that if this third spinor is indeed Killing, then it is also the case for the two independent spinors on a given Lorentz orbit in which it can be decomposed:

$$\xi_3 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} 1 + \begin{pmatrix} -d^* \\ -c^* \\ b^* \\ a^* \end{pmatrix} e_{12} = \begin{pmatrix} a \\ 0 \\ 0 \\ d \end{pmatrix} 1 + \begin{pmatrix} -d^* \\ 0 \\ 0 \\ a^* \end{pmatrix} e_{12} + \begin{pmatrix} 0 \\ b \\ c \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 0 \\ -c^* \\ b^* \\ 0 \end{pmatrix} e_{12}\,, \tag{4.32}$$

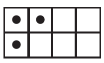
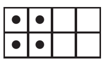
since the coefficients (a,d) and (b,c) do not mix. Furthermore, the equations for the pair (a,d) are exactly the same as the ones for (b,c), therefore if

$$\xi_3 = \begin{pmatrix} a \\ 0 \\ 0 \\ d \end{pmatrix} 1 + \begin{pmatrix} -d^* \\ 0 \\ 0 \\ a^* \end{pmatrix} e_{12} \tag{4.33}$$

is a Killing spinor, then automatically

$$\xi_4 = \begin{pmatrix} 0 \\ a \\ d \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 0 \\ -d^* \\ a^* \\ 0 \end{pmatrix} e_{12} \tag{4.34}$$

will also be Killing. Thus

 $\Rightarrow$ 

For the sake of definiteness, let us choose $a = i$ and $d = 0$. One then obtains that a new scalar field, in particular ϕ^{24}, decouples, and the equations reduce to

$$\begin{aligned}
F^{1\bar{1}} + F^{2\bar{2}} - \left[\phi^{12},\phi^{34}\right] &= 0\,,\\
F^{\bar{1}\bar{2}} &= 0\,,
\end{aligned}$$

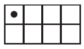
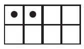
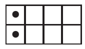
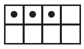
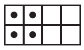
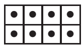
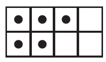
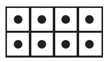
	One susy
 	Two susys
	Three susys (F selfdual)
	Four susys
	Eight susys: instanton

Table 2. Possible supersymmetric configurations for Killing spinors of the same chirality.

$$\begin{aligned} D^1\phi^{34} &= D^2\phi^{34} = 0\,, \\ D^{\bar{1}}\phi^{12} &= D^{\bar{2}}\phi^{12} = 0\,, \\ \left[\phi^{23},\phi^{AB}\right] = \left[\phi^{14},\phi^{AB}\right] = \left[\phi^{13},\phi^{AB}\right] = \left[\phi^{24},\phi^{AB}\right] &= 0\,, \\ D^\mu\phi^{23} = D^\mu\phi^{14} = D^\mu\phi^{13} = D^\mu\phi^{24} &= 0\,. \end{aligned} \tag{4.35}$$

For a generic spinor, it is another combination of the scalar fields that would decouple. Finally, adding one more right Killing spinor to the $(2,2,0,0)$ configuration immediately leads to the instanton solution, i.e. the decoupling of all scalar fields and selfdual field strength, with eight supersymmetries preserved:

 $\Rightarrow$ 

The possible configurations with Killing spinors of the same chirality are summarized in table 2.

4.2.2 Killing spinors with different chiralities

We again start with the equations (4.17) imposed by the first Killing spinor and from there we proceed methodically to take into account the constraints coming from additional supersymmetries.

$\mathbf{(1,0,0,0)\rightarrow(1,0,1,0)\rightarrow(1,0,2,0)=(2,0,2,0).}$ Combining the equations (4.17) from the first spinor with those arising from plugging (4.12) in (4.16), one gets

$$(1,0,1,0) \quad \Leftrightarrow \quad \left\{ \begin{array}{lll} F^{1\bar{1}} &= [\phi^{12},\phi^{34}]+[\phi^{13},\phi^{24}]\,, & \\ F^{1\bar{2}}+[\phi^{12},\phi^{13}] &= 0\,, & F^{2\bar{2}}=0\,, \\ [\phi^{23},\phi^{AB}] &= [\phi^{14},\phi^{AB}]=0 & \forall\, A,B\,, \\ D^\mu\phi^{23} &= D^\mu\phi^{14}=0\,, & \\ [\phi^{12},\phi^{13}] &= -[\phi^{12},\phi^{24}]\,, & \\ [\phi^{13},\phi^{34}] &= -[\phi^{24},\phi^{34}]\,, & \\ (D^2-D^{\bar{2}})\phi^{AB} &= 0\,. & \end{array} \right. \tag{4.36}$$

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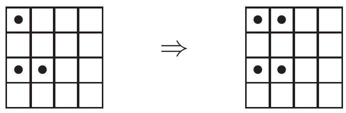
Therefore the field ϕ^{14} decouples, while subsequent Killing spinors could never lead to a non-trivial selfdual solution. When adding a generic left spinor (4.11) to $(1,0,1,0)$, one sees similarly to (4.31) and (4.32) that the two spinors lying on the Lorentz orbits of (4.12) and (4.13) respectively into which it decomposes are also Killing, because the (e,h) and (f,g) components do not mix. Let us separate the two cases and continue with $(1,0,2,0)$, thus implementing the constraints coming from

$$\bar{\xi}_3 = \begin{pmatrix} e \\ 0 \\ 0 \\ h \end{pmatrix} e_1 + \begin{pmatrix} h^* \\ 0 \\ 0 \\ -e^* \end{pmatrix} e_2 \,. \tag{4.37}$$

One verifies that these constraints automatically imply that

$$\xi_4 = \begin{pmatrix} e \\ 0 \\ 0 \\ -h^* \end{pmatrix} 1 + \begin{pmatrix} h \\ 0 \\ 0 \\ e^* \end{pmatrix} e_{12} \tag{4.38}$$

belonging to the Lorentz orbit of (4.3) is a Killing spinor, therefore



$\mathbf{(1,0,0,0) \to (1,0,1,0) \to (1,0,1,1) = (1,1,1,1) \to (2,1,1,1) = (2,2,2,2) \to (3,2,2,2) = (4,4,4,4).}$ Starting from $(1,0,1,0)$, one would like to combine the constraints (4.36) with those coming from a spinor of the form

$$\bar{\xi}_3 = \begin{pmatrix} 0 \\ f \\ g \\ 0 \end{pmatrix} e_1 + \begin{pmatrix} 0 \\ g^* \\ -f^* \\ 0 \end{pmatrix} e_2 \,. \tag{4.39}$$

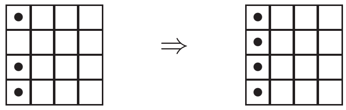
Actually, this form can be simplified using the subgroup stabilizing (4.3) and (4.12), composed of the 6 generators $S_1, \ldots, S_3$ and $S_8, \ldots, S_{10}$. (The latter, of course, do not act on $\bar{\xi}_3$). With $f = B\exp(2i\varphi_B)$ and $g = C\exp(2i\varphi_C)$, $\bar{\xi}_3$ can be brought to

$$\bar{\xi}_3 = \begin{pmatrix} 0 \\ B\sqrt{1+C^2/B^2} \\ 0 \\ 0 \end{pmatrix} e_1 + \begin{pmatrix} 0 \\ 0 \\ -B\sqrt{1+C^2/B^2} \\ 0 \end{pmatrix} e_2 \tag{4.40}$$

by acting with $\exp[(\varphi_B + \varphi_C)S_1] \cdot \exp[(\operatorname{arctg} C/B)S_2] \cdot \exp[(\varphi_B - \varphi_C)S_1]$. Equations (4.36) are then supplemented by the conditions

$$\begin{aligned} D^1\phi^{34} + D^{\bar{2}}\phi^{24} &= 0\,, \\ D^{\bar{1}}\phi^{24} - D^2\phi^{34} &= 0\,, \\ D^1(\phi^{13} + \phi^{24}) &= 0\,, \\ \left[\phi^{13}, \phi^{24}\right] &= 0\,. \end{aligned} \tag{4.41}$$

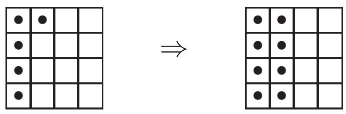
One may then check that the susy equations for the Killing spinor (4.10) are satisfied as a consequence of (4.36) and (4.41), yielding



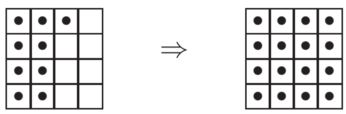
As a next step, let us impose an additional Killing spinor in one of the Lorentz orbits, say the first without loss of generality. We thus take it of the form (4.9), in such a way that it be linearly independent of the first one. The equations coming out by combining the new constraints with the former ones are not really enlightening, so we just mention the expression of the field strength:

$$\begin{aligned} F^{1\bar{1}} &= \frac{2i}{d^*}\mathrm{Im}(a)\left[\phi^{12}, \phi^{24}\right]\,, \qquad F^{2\bar{2}} = 0\,, \\ F^{\bar{1}2} &= F^{1\bar{2}} = \frac{d}{d^*}\left[\phi^{12}, \phi^{24}\right]\,. \end{aligned} \tag{4.42}$$

However, more importantly, one can check that after having imposed the conditions for the latter $(2,1,1,1)$ configuration, the following spinors are Killing: (4.34) with $a = i$, (4.37) with $h = -d^*\mathrm{Im}(e)$ and (4.39) with $g = -d^*\mathrm{Im}(f)$, and hence



At last, adding the constraints of one more Killing spinor belonging to one of the four Lorentz orbits, say the first, leads to the vacuum solution preserving all 16 supersymmetries:



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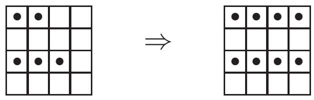
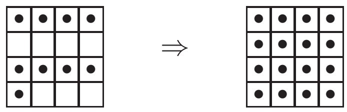
$(2,0,2,0) \rightarrow (2,0,3,0) = (4,0,4,0) \rightarrow (4,0,4,1) = (4,4,4,4)$. Supplementing the constraints of $(2,0,2,0)$ by the one originating from one more left Killing spinor on the same Lorentz orbit, configuration $(2,0,3,0)$ leads to the vanishing of the field strength, all covariant derivatives and all commutators but two, $[\phi^{12},\phi^{34}] = [\phi^{13},\phi^{24}]$. These restrictions imply the supersymmetry equations for any spinor on the orbits of (4.3) or (4.12), and consequently



Finally, adding any spinor to $(4,0,4,0)$ leads to the vanishing of the last commutators, and thus to the vacuum solution preserving all 16 supersymmetries:



$(1,0,0,0) \rightarrow (1,0,0,1) \rightarrow (1,0,0,2) \rightarrow (1,0,0,3) \rightarrow (1,0,0,4) \rightarrow (2,0,0,4)$. Let us now re-start with the second Killing spinor of opposite chirality on the other orbit, thus of the form (4.13). Combining its constraints with those of the first Killing spinor yields for the field strength

$$F^{1\bar{1}} = \left[\phi^{12},\phi^{34}\right] , \qquad F^{2\bar{2}} = -\left[\phi^{13},\phi^{24}\right] , \tag{4.43}$$
$$F^{\bar{1}\bar{2}} = -\left[\phi^{24},\phi^{34}\right] , \qquad F^{\bar{1}2} = \left[\phi^{13},\phi^{34}\right] ,$$

or equivalently

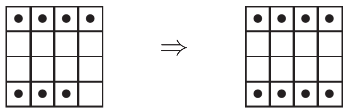
$$F - \phi \wedge \phi = 0 . \tag{4.44}$$

One can subsequently look for the existence of additional Killing spinors belonging for example to the Lorentz orbit of the second Killing spinor. Naturally, after the second one, all spinors cannot be further simplified and have to be taken generic, of the form (4.39). Again, the equations involving the covariant derivatives of the scalar fields are rather involved and do not tell much, so we will focus on the field strength. From $(1,0,0,2)$, one observes that $F^{\bar{1}2} = 0$, although F is not yet anti-selfdual. For the configuration $(1,0,0,3)$, as expected from section 4.2.1, the field strength becomes anti-selfdual (as the for the configuration $(0,0,0,3)$), while the remaining component $F^{\bar{1}\bar{2}}$ is now determined in terms of the commutator of scalars:

$$F^{1\bar{1}} = F^{2\bar{2}} = -[\phi^{13},\phi^{24}] , \qquad F^{\bar{1}2} = 0 , \tag{4.45}$$
$$F^{\bar{1}\bar{2}} = -[\phi^{24},\phi^{34}] .$$

For $(1,0,0,4)$, we get of course something very similar to (4.29), with the difference that the field strength is anti-selfdual instead of selfdual, and with the additional information that can be seen in (4.45) that the components of F are related to commutators of scalar fields (which was not the case in (4.29)). Adding one more Killing spinor on the orbit of the first spinor further yields $F^{\bar{1}\bar{2}} = -[\phi^{24}, \phi^{34}] = 0$.

$(1,0,0,2) \rightarrow (2,0,0,2) \rightarrow (3,0,0,2)/(2,0,0,3) \rightarrow (3,0,0,3) \rightarrow (4,0,0,3) = (4,0,0,4)$. The last cases that haven't yet been explored or that are not a consequence of what we have seen up to now consist in adding to $(1,0,0,2)$ a Killing spinor in the Lorentz orbit of the first representative. The configuration $(2,0,0,2)$ will not be (anti-)selfdual, but has $F^{\bar{1}2} = F^{12} = 0$. Next, as expected, $(3,0,0,2)$ is selfdual while $(2,0,0,3)$ is anti-selfdual. As a consequence, $(3,0,0,3)$ has vanishing field strength. Finally, one can see using a basis for elements on the first and fourth orbit that



The classification of supersymmetric backgrounds of $\mathcal{N} = 4$ SYM theory is summarized in table 3.

5 Susy variations and equations of motion

The Killing spinor equations arising from setting to zero the supersymmetry variations (2.5) are first order, and one could ask whether they imply the second order equations of motion (2.6)–(2.9). In supergravity, this is not always the case: The Killing vector constructed as a bilinear from the Killing spinor can be either timelike or lightlike. One can show that in the former case, the Killing spinor equations, together with the Bianchi identities and the Maxwell equations, do entail the Einstein equations, whereas in the null case, one of the Einstein equations must be additionally imposed by hand (cf. e.g. [21]).

We first focus on purely bosonic configurations. Let us consider the conditions (4.17) imposed by the existence of a chiral Killing spinor. One would like to check whether these imply

$$\begin{cases} D^\nu F_{\nu\mu} - \frac{1}{2}\left[\bar{\phi}_{AB}, D_\mu \phi^{AB}\right] = 0\,, \\ D^2 \phi^{AB} - \frac{1}{2}\left[\bar{\phi}_{CD}, \left[\phi^{AB}, \phi^{CD}\right]\right] = 0\,. \end{cases} \tag{5.1}$$

A first observation is that one the scalar fields, here ϕ^{14}, does not appear in the susy equations (except through the fact that it commutes with ϕ^{23}). Therefore its equations of motion are certainly not satisfied by virtue of the susy equations, and the former will have to be imposed by hand. However, all the other equations of motion will automatically hold, using only (4.17), the Jacobi identity and the Bianchi identities for the gauge field, as we illustrate now.

1	2	3	4	5	6	8	16
(1,0,0,0)	(2,0,0,0)	(3,0,0,0)	(4,0,0,0)	(3,0,0,2)	(3,0,0,3)	(4,4,0,0)	(4,4,4,4)
	(1,1,0,0)	(2,0,0,1)	(2,2,0,0)	(4,0,0,1)	(4,0,0,2)	(4,0,4,0)	
	(1,0,1,0)		(2,0,2,0)			(2,2,2,2)	
	(1,0,0,1)		(1,1,1,1)			(4,0,0,4)	
			(2,0,0,2)				
			(3,0,0,1)				

Table 3. Classification of purely bosonic supersymmetric configurations of $\mathcal{N} = 4$ SYM theory, with Killing spinors of definite chiralities. The first line indicates the number n of supersymmetries. Notice that there are no backgrounds with $n = 7$ or $9 \leq n \leq 15$. An arbitrary configuration (n_1, n_2, n_3, n_4) if not present in the table can be shown to be equivalent to one of the above using the analysis of the previous section. As an example, let us consider $(2, 0, 2, 1)$. We have seen that $(1, 0, 1, 1) = (1, 1, 1, 1)$ (the equality sign meaning "implies"), therefore $(2, 0, 2, 1) = (2, 1, 2, 1)$. But we also observed that $(2, 1, 1, 1) = (2, 2, 2, 2)$. Therefore $(2, 0, 2, 1) = (2, 2, 2, 2)$.

Consider the first equation of (5.1) for e.g. $\mu = 1$. By expanding it, using the definition $\bar{\phi}_{AB} = \frac{1}{2}\epsilon_{ABCD}\phi^{CD}$ for the duals and the fact that ϕ^{23} decouples, one gets

$$\begin{aligned} &D^{\bar{1}}F^{1\bar{1}} + D^2 F^{\bar{2}\bar{1}} + D^{\bar{2}}F^{2\bar{1}} \\ &\quad -[\phi^{34}, D_1\phi^{12}] - [\phi^{42}, D_1\phi^{13}] - [\phi^{31}, D_1\phi^{24}] - [\phi^{12}, D_1\phi^{34}] \stackrel{?}{=} 0\,. \end{aligned} \tag{5.2}$$

Now use the Bianchi identity

$$D^{\bar{2}}F^{2\bar{1}} = D^2 F^{\bar{2}\bar{1}} + D^{\bar{1}}F^{2\bar{2}}\,, \tag{5.3}$$

to replace the term $D^{\bar{1}}F^{1\bar{1}} + D^2 F^{\bar{2}\bar{1}} + D^{\bar{2}}F^{2\bar{1}}$ in (5.2) by $2D^2F^{\bar{2}\bar{1}} + D^{\bar{1}}(F^{1\bar{1}} + F^{2\bar{2}})$ and finally the two first equations of (4.17) to eliminate the field strengths in terms of commutators of scalar fields, to see that they exactly cancel the scalar field part in (5.2). The other components are satisfied in the same way.

We now turn to the second equation of (5.1) and show explicitly how things combine for $A = 1$, $B = 3$. (For all other index combinations except $A = 1$, $B = 4$, the proof is analogous). First the $D_\mu D^\mu \phi^{13}$ term is rewritten as $2(D^1D^{\bar{1}} + D^2D^{\bar{2}})\phi^{13} - [F^{1\bar{1}} + F^{2\bar{2}}, \phi^{13}]$ which, using the third equation of (4.17), as well as the complex conjugate of the fourth, boils down to $-2[F^{12}, \phi^{13}] - [F^{1\bar{1}} + F^{2\bar{2}}, \phi^{13}]$. Using then the first and the complex conjugate of the second equation of (4.17), one arrives at a sum of 3 multicommutator terms involving only ϕ^{12}, ϕ^{13} and ϕ^{34} which vanishes by virtue of the Jacobi identity.

6 Final remarks

In this work we have discussed the classification of supersymmetric solutions of euclidean $\mathcal{N} = 4$ SYM theory in 4 dimensions. We have displayed the equations they satisfy in the spinorial geometry language. The equations one gets by imposing the existence of a single Killing spinor can be obtained by dimensional reduction from 8 to 4 dimensions of the

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octonionic instanton equations, much like the Hitchin equations arise by reduction of the selfduality equations from 4 to 2 dimensions. The system of equations may be rephrased compactly in terms of a selfduality-like condition for a complex connection. We next wrote down the equations arising by imposing the existence of more Killing spinors (focusing on Killing spinors with definite chiralities). Their analysis led to the conclusion that not all fractions of the maximal number of supersymmetries are allowed. In particular, there are no bosonic configurations preserving 7 supersymmetries (because this automatically implies the existence of an 8th Killing spinor) nor 9 to 15 (that would imply 16). This is reminiscent of more familiar set-ups, for instance from general relativity (no metrics with 8 or 9 Killing vectors in 4 dimensions), and supergravity (e.g. no BPS solutions in eleven-dimensional sugra preserving 31 supersymmetries [22]).

We focused on purely bosonic configurations, but solutions with non-trivial fermionic fields are of course not excluded and are certainly worth studying. Also, it would be desirable to work out explicit solutions to these various sets of equations and verify whether their corresponding on-shell action is finite. In the affirmative, the corresponding configurations would represent instantons possibly including non-trivial scalar and fermion fields profiles, which as of today are not known in closed form.[10] The interest in instanton effects in $\mathcal{N}=4$ SYM is at least twofold. On the one hand, the theory is believed to be self-dual [25], a statement entailing the complete effective action including all instanton and anti-instanton effects. On the other hand, instantons have provided some of the most striking tests of the AdS/CFT correspondence. From this point of view, instantons with topological charge k in $\mathcal{N}=4$ SYM with SU(N) gauge group are obtained by adding k D-instantons (D(-1) branes) to the stack of N $D3$ branes [27–30] (for unoriented D-instantons, one starts with D3-branes on top of an orientifold 3-plane, the gauge group of the $\mathcal{N}=4$ SYM theory becoming Sp(N) or SO(N) depending on the charge of the O3-plane [26]). In the low energy supergravity limit where computations can mostly be performed (see however [5], section 18.3 for a discussion beyond sugra, in the BMN limit), D-instantons arise as non-trivial solutions of the Euclidean field equations. The classical type IIB supergravity action in $AdS_5 \times S^5$ can take these into account by incorporating the effect of the infinite tower of massive string excitations on the dynamics of the massless modes. In the case of minimal correlators/AdS amplitudes, it turns out there is a perfect agreement between instanton contributions to SYM correlation functions and D-instanton induced supergravity amplitudes (see [31–33], or [5], sections 15-18, [12] for reviews). Of course, this agreement is striking since the computations on the field theory side are done at weak coupling, and indicates that the corresponding correlators are protected from quantum corrections. On the gauge theory side, the latter computations are performed in a particular instanton background. The latter is generated from the self-dual configuration, starting from the YM instanton and solving iteratively the full set of coupled equations to get an approximated truncated solution by retaining terms only up to a certain power of the coupling constant, which is enough to compute correlators in the semi-classical approximation, see e.g. section 14 of [5]). If new finite-action solutions would appear to exist, in particular with non (anti-

[10]Note that it has been argued that such a solution may not exist for generic gauge group [13].

)selfdual field strength, it would certainly be a very interesting problem to compute their contribution to correlation functions and match it with a dual supergravity computation.

Finally, the AdS/CFT correspondence has also allowed to give a string theory interpretation of SYM states. For example, it is known that monopoles and dyons are dual to D-strings and bound states of D-strings and fundamental strings, respectively, between different D3-branes. In the same spirit, 1/4 BPS states in $\mathcal{N} = 4$ SYM with a gauge group SU(3) have been shown to correspond to three-pronged strings connecting three D3-branes [23]. It would therefore be really interesting to identify to which configurations on the gravity side these various supersymmetric solutions (or their counterparts in the lorentzian theory) are mapped through the AdS/CFT correspondence. The strategy of the present analysis is of course not limited to four-dimensional $\mathcal{N} = 4$ SYM, and could be applied to any supersymmetric gauge theory. An example of particular interest are the superconformal three-dimensional Chern-Simons theories recently proposed in the context of the AdS_4/CFT_3 corespondence [24]. We hope to return to these questions in future works.

Acknowledgments

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A Conventions

In this appendix, we collect some of our conventions and notations in relation to spinors and Clifford algebras based on [8].

The 't Hooft symbols are defined as

$$\begin{aligned} \eta_{a\mu\nu} &= \epsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{4\mu} \,, \\ \bar{\eta}_{a\mu\nu} &= \epsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{4\mu} \,, \end{aligned} \tag{A.1}$$

with $a = 1, 2, 3$ and $\mu, \nu = 1, 2, 3, 4$. The three matrices η_a are selfdual, while the $\bar{\eta}_a$ are anti-selfdual,

$$\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\eta_{a\rho\sigma} = \eta_{a\mu\nu} \,, \qquad \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\bar{\eta}_{a\rho\sigma} = -\bar{\eta}_{a\mu\nu} \,, \qquad (\epsilon_{1234} = 1) \,, \tag{A.2}$$

and together they form a basis for the 4×4 antisymmetric matrices. Moreover, they satisfy the relations

$$\begin{aligned} [\eta_a, \eta_b] &= -2\epsilon_{abc}\eta_c \,, & [\bar{\eta}_a, \bar{\eta}_b] &= -2\epsilon_{abc}\bar{\eta}_c \,, \\ \{\eta_a, \eta_b\} &= -2\delta_{ab} \,, & \{\bar{\eta}_a, \bar{\eta}_b\} &= -2\delta_{ab} \,, \\ [\eta_a, \bar{\eta}_b] &= 0 \,. \end{aligned} \tag{A.3}$$

When discussing spinors in 6 dimensions, we will use the notation

$$\vec{\eta} = (\eta_1, \eta_2, \eta_3) \,, \qquad \vec{\bar{\eta}} = (\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3) \,, \tag{A.4}$$

understood as three-component vectors of 4×4 matrices.

We use the following representation of the four-dimensional euclidean Clifford algebra:

$$\gamma_\mu = \begin{pmatrix} 0 & -i\sigma_\mu \\ i\bar{\sigma}_\mu & 0 \end{pmatrix} \,, \tag{A.5}$$

with

$$\sigma_\mu = (\vec{\tau}, i) \,, \qquad \bar{\sigma}_\mu = (\vec{\tau}, -i) \,, \qquad \mu = 1, \dots, 4 \,, \tag{A.6}$$

$\vec{\tau}$ denoting the three Pauli matrices. In this representation the Spin(4) generators on four-component Dirac spinors are

$$\gamma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar{\sigma}_{\mu\nu} \end{pmatrix} \,, \tag{A.7}$$

$$\sigma_{\mu\nu} = \frac{1}{2} \left(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu \right) = i \bar{\eta}_{a\mu\nu} \tau^a \,, \tag{A.8}$$

$$\bar{\sigma}_{\mu\nu} = \frac{1}{2} \left(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu \right) = i \eta_{a\mu\nu} \tau^a \,, \tag{A.9}$$

where the relationship to the 't Hooft symbols has been made explicit. The matrices $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are the generators of the two inequivalent pseudo-real irreducible representations of Spin(4) acting on two-component Weyl spinors ψ^α and $\bar{\chi}_{\dot{\alpha}}$ respectively. Indices $\alpha, \dot{\alpha} = 1, 2$ are raised and lowered according to the north-west convention

$$\epsilon^{\alpha\beta} \psi_\beta = \psi^\alpha \,, \qquad \bar{\psi}^{\dot{\beta}} \epsilon_{\dot{\beta}\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}} \,, \tag{A.10}$$

where the antisymmetric invariant tensor ϵ is defined by

$$\epsilon_{12} = 1 \,, \qquad \epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} \,, \qquad \epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta} \,, \qquad \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\alpha\beta} \,, \qquad \epsilon_{\dot{\beta}\dot{\alpha}} = -\epsilon_{\dot{\alpha}\dot{\beta}} \,.$$

In 6 dimensions, one defines the four by four matrices

$$\Sigma^a = \left(-i\eta_1, \eta_2, \eta_3, i\vec{\bar{\eta}} \right) \,, \qquad \bar{\Sigma}^a = \left(i\eta_1, -\eta_2, -\eta_3, i\vec{\bar{\eta}} \right) \,, \tag{A.11}$$

with elements $\Sigma^{a,AB}$ and $\bar{\Sigma}^a_{AB}$, where indices $a = 1, \dots, 6$ are raised with the flat Minkowski metric in 5+1 dimensions. The matrices

$$\hat{\gamma}_a = \begin{pmatrix} 0 & \Sigma_a \\ \bar{\Sigma}_a & 0 \end{pmatrix} \tag{A.12}$$

form a representation of the Clifford algebra $Cl(5,1)$. The corresponding representation of $\mathrm{Spin}(5,1)$ is again reducible into two pseudo-real inequivalent representations, with right Weyl spinors transforming with $\Sigma_{ab} = \frac{1}{2}(\Sigma_a \bar{\Sigma}_b - \Sigma_b \bar{\Sigma}_a)$ and left Weyl spinors with $\bar{\Sigma}_{ab} = \frac{1}{2}(\bar{\Sigma}_a \Sigma_b - \bar{\Sigma}_b \Sigma_a)$, i.e.,

$$\hat{\gamma}_{ab} = \begin{pmatrix} \Sigma_{ab} & 0 \\ 0 & \bar{\Sigma}_{ab} \end{pmatrix} \,. \tag{A.13}$$

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